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Bijections for Partition Identities

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We consider partition-identity bijections that can be constructed from sieve-equivalent families whose defining multisets are pairwise disjoint. We give a linear algorithm that constructs these bijections. We show that under these conditions, the new algorithm, the Garsia–Milne–Remmel algorithm, and the Gordon algorithm are equivalent, i.e., that they produce the same bijection. © 1988 Academic Press, Inc.

INTRODUCTION

In [4], Remmel gave a bijective proof of a general partition theorem via the involution principle of Garsia and Milne [2]. It was an extension of a non-bijective theorem of Cohen [1], who used, in his proof, the principle of inclusion/exclusion. Subsequently, Gordon [3] published an extension of Remmel's theorem, but used a recursive algorithm for its bijective proof.

In general, the use of the involution principle and/or Gordon's algorithm results in long calculations and complicated bijections. Yet Remmel was able to show that the bijections which resulted from special cases of his theorem were the same, in fact, as classical bijections due to Glaisher, Subbaro, Andrews, and others. Its effect was to unify many of the classical results in the theory of partition identities.

Here is an example of one of these classical partition identities due to Euler: the number of partitions whose parts are odd equals the numbers of partitions whose parts are distinct. The proof depends on the fact that every integer l has a unique binary representation

$$l = 2^a + 2^b + 2^c + \cdots.$$

So if a partition π of n with odd parts is written

$$\begin{aligned} n &= l_1 \cdot 1 + l_2 \cdot 3 + l_3 \cdot 5 + \cdots \\ &= (2^{a_1} + 2^{b_1} + \cdots)\, 1 + (2^{a_2} + 2^{b_2} + \cdots)\, 3 + (2^{a_3} + \cdots)\, 5 + \cdots \end{aligned}$$

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then π's bijective mate is

$$2^{a_1}, 2^{b_1}, ..., 3 \cdot 2^{a_2}, 3 \cdot 2^{b_2}, ..., 5 \cdot 2^{a_3}, ...$$

Euler's identity is really a representative example of the class of partition identities orginally considered in Cohen's paper, the disjoint case. In this paper, we show that in the disjoint case there is an efficient algorithm, patterned after Glaisher's proof above, that produces the same bijections as the Garsia–Milne–Remmel and Gordon algorithms.

Notation and Definitions

We think of a partition, π, as a multiset of positive integers whose sum is n, i.e., $\pi = \{1^{r_1}, 2^{r_2}, ..., p^{r_p}\}$, where i^{r_i} means that π contains r_i copies of the integer i. Given two partitions

$$\pi = \{1^{r_1}, 2^{r_2}, ..., k^{r_k}\} \qquad \text{and} \qquad \lambda = \{1^{q_1}, 2^{q_2}, ..., k^{q_k}\}$$

we define

$$\pi \cup \lambda = \{1^{p_1}, 2^{p_2}, ..., k^{p_k}\}, \qquad \text{where} \quad p_i = \max\{r_i, q_i\}.$$

Given two lists of multisets

$$\mathfrak{L}_1 = \{a_1, a_2, ..., a_m\} \qquad \text{and} \qquad \mathfrak{L}_2 = \{b_1, b_2, ..., b_m\}$$

we say that a partition, π, has the a_i-property if $a_i \subseteq \pi$, and has the b_i-property if $b_i \subseteq \pi$. Let $A_i = \{\pi \vdash n \mid a_i \subseteq \pi\}$ and $B_i = \{\pi \vdash n \mid b_i \subseteq \pi\}$.

If

$$\left|\bigcup_{i \in S} a_i\right| = \left|\bigcup_{i \in S} b_i\right| \qquad \text{for all} \quad S \subseteq M = \{1, 2, ..., m\},$$

then in the case of partitions

$$\left|\bigcap_{i \in S} A_i\right| = \left|\bigcap_{i \in S} B_i\right| \qquad \text{for all} \quad S \subseteq M$$

and we say the two families

$$\mathfrak{A} = \{A_1, A_2, ..., A_m\}$$

and

$$\mathfrak{B} = \{B_1, B_2, ..., B_m\}$$

are sieve-equivalent [5].

Let $A_0 = \{\pi \vdash n \mid a_i \nsubseteq \pi \ \forall i \in M\}$ and $B_0 = \{\pi \vdash n \mid b_i \nsubseteq \pi \ \forall i \in M\}$.

If $a_i \cap a_j = \varnothing$ and $b_i \cap b_j = \varnothing$ $(i \neq j)$, we say that the defining multisets of $\mathfrak{A}$ and $\mathfrak{B}$ are pairwise disjoint.

Example. Consider the list of multisets

$$a_1, a_2, a_3, ... = \{11\}, \{22\}, \{33\}, ...$$

and

$$b_1, b_2, b_3, ... = \{2\}, \{4\}, \{6\}, ...$$

If $n = 6$ then

$$A_1 = \{411, 3111, 2211, 21111, 111111\}$$
$$B_1 = \{42, 321, 222, 2211, 21111\}$$
$$A_2 = \{222, 2211\}, \qquad A_3 = \{33\}, \qquad A_{12} = A_1 \cap A_2 = \{2211\}$$
$$B_2 = \{42, 411\}, \qquad B_3 = \{6\}, \qquad B_{12} = B_1 \cap B_2 = \{42\}$$
$$A_0 = \{6, 51, 42\} \qquad \text{and} \qquad B_0 = \{51, 33, 111111\}.$$

Using this notation we can now state Remmel's theorem.

Theorem (Remmel [4]). *Suppose* $\mathfrak{A} = \{A_1, A_2, ..., A_m\}$ *and* $\mathfrak{B} = \{B_1, B_2, ..., B_m\}$ *are sieve-equivalent families of partitions of n. Then* $|A_0| = |B_0|$.

In this paper, we restrict our attention to the case where $\mathfrak{A}$ and $\mathfrak{B}$ are defined by pairwise disjoint multisets $\{a_i\}_i$ and $\{b_i\}_i$. This is exactly the case Cohen considered, and includes our example of Euler's identity.

A map that exchanges an a_i-multiset for a b_i-multiset is written

$$f_i(\pi) = \pi - a_i + b_i$$
$$f_i^{-1}(\pi) = \pi - b_i + a_i,$$

where subtraction and addition are multiset operations. If $a_i \subseteq \pi$ (resp.

$b_i \subseteq \pi$) then $f_i(\pi)$ (resp. $f_i^{-1}(\pi)$) is said to be consistent. Similarly, we say that the product

$$f_{q(k)}^{\varepsilon(k)} f_{q(k-1)}^{\varepsilon(k-1)} \cdots f_{q(1)}^{\varepsilon(1)}(\pi),$$

where $\varepsilon(i) = +1$ or -1, is consistent if for each $j = 1, 2, ..., k$ the product

$$f_{q(j)}^{\varepsilon(j)} f_{q(j-1)}^{\varepsilon(j-1)} \cdots f_{q(1)}^{\varepsilon(1)}(\pi)$$

is consistent.

Remark. If $\mathfrak{A}$ and $\mathfrak{B}$ are sieve-equivalent, then $f_S(\pi)$ is weight preserving.

We include a sample of each of the three algorithms, the Garsia–Milne–Remmel, the Gordon, and the author's, for comparison.

EXAMPLE. Euler's Identity. The defining multisets are

$$a_i = \{ii\} \qquad \text{and} \qquad b_i = \{2i\}.$$

Let $n = 8$. Let $\pi_0 = 11111111$, a partition with odd parts. We seek the bijective mate of π_0, $\tilde{\pi}$, a partition with distinct parts.

Case 1. Garsia–Milne–Remmel Algorithm.

The GMRA is a sequence of α, β, f, and f^{-1} maps operating on ordered pairs (π, s), where π is a partition and S is a subset of M, and defined as follows:

$$f(\pi, S) = \left(\pi - \bigcup_{i \in S} a_i + \bigcup_{i \in S} b_i, S\right)$$

$$f^{-1}(\pi, S) = \left(\pi - \bigcup_{i \in S} b_i + \bigcup_{i \in S} a_i, S\right)$$

$$\alpha(\pi, S) = \begin{cases} (\pi, S - a_\pi) & \text{if} \quad a_\pi \in S \\ (\pi, S \cup a_\pi & \text{if} \quad a_\pi \notin S \end{cases}$$

$$\beta(\pi, S) = \begin{cases} (\pi, S - b_\pi) & \text{if} \quad b_\pi \in S \\ (\pi, S \cup b_\pi) & b_\pi \notin S, \end{cases}$$

where

$$a_\pi = \max\{i \mid a_i \subseteq \pi\}$$

and

$$b_\pi = \max\{i \mid b_i \subseteq \pi\}.$$

If $\pi_0 = 11111111$ we have

$$\begin{aligned}
(11111111, \phi) &\xrightarrow{f} (11111111, \phi)\\
(1111112, 1) &\xleftarrow{f^{-1}} (11111111, 1)\\
(1111112, \phi) &\xrightarrow{f} (1111112, \phi)\\
(111122, 1) &\xleftarrow{f^{-1}} (1111112, 1)\\
(111122, \phi) &\xrightarrow{f} (111122, \phi)\\
(11114, 2) &\xleftarrow{f^{-1}} (111122, 2)\\
(11114, \phi) &\xrightarrow{f} (11114, \phi)\\
(1124, 1) &\xleftarrow{f^{-1}} (11114, 1)\\
(1124, 12) &\xrightarrow{f} (111122, 12)\\
(11222, 1) &\xleftarrow{f^{-1}} (111122, 1)\\
(11222, \phi) &\xrightarrow{f} (11222, \phi)\\
(1124, 2) &\xleftarrow{f^{-1}} (11222, 2)
\end{aligned}$$

$$\begin{aligned}
(1124, \phi) &\xrightarrow{f} (1124, \phi)\\
(224, 1) &\xleftarrow{f^{-1}} (1124, 1)\\
(224, 12) &\xrightarrow{f} (11222, 12)\\
(2222, 1) &\xleftarrow{f^{-1}} (11222, 1)\\
(2222, \phi) &\xrightarrow{f} (2222, \phi)\\
(224, 2) &\xleftarrow{f^{-1}} (2222, 2)\\
(224, \phi) &\xrightarrow{f} (224, \phi)\\
(44, 2) &\xleftarrow{f^{-1}} (224, 2)\\
(44, \phi) &\xrightarrow{f} (44, \phi)\\
(8, 4) &\xleftarrow{f^{-1}} (44, 4)\\
(8, \phi) &\xrightarrow{f} (8, \phi) \mid
\end{aligned}$$

a total of 46 steps.

Case 2. Gordon Algorithm.

The Gordon algorithm is a sequence of f_ϕ maps alternating with $h_{T,0}^{-1}$ maps, where

$$h_{T,0}^{-1}\colon\ B_{T,0} \to A_{T,0},$$

where $B_{T,0}$ ($A_{T,0}$) is the set of partitions of n that contain exactly b_i-multisets for all $i \in S$ (resp. a_i-multisets for all $i \in S$). One uses recursive calls to the algorithm to construct the $h_{T,0}$ maps when needed.

If $\pi_0 = 11111111$ then

$$
\begin{array}{lcll}
11111111 & \xrightarrow{f_\emptyset} & 11111111 & \} \; h^{-1}_{1,0} \\
& \swarrow f_1^{-1} & & \\
1111112 & \xrightarrow{f_\emptyset} & 1111112 & \} \; h^{-1}_{1,0} \\
& \swarrow f_1^{-1} & & \\
111122 & \xrightarrow{f_\emptyset} & 111122 & \} \; h^{-1}_{12,0} \\
& \swarrow f_{12}^{-1} & & \\
11124 & \xrightarrow{f_\emptyset} & 1124 & \multirow{3}{*}{$\} \; h^{-1}_{1,0}$} \\
& \swarrow f_1^{-1} & & \\
224 & \xrightarrow{f_{12}} & 11222 & \\
& \swarrow f_1^{-1} & & \\
2222 & \xrightarrow{f_\emptyset} & 2222 & \multirow{5}{*}{$\} \; h^{-1}_{2,0}$} \\
& \swarrow f_2^{-1} & & \\
224 & \xrightarrow{f_{12}} & 11222 & \\
& \swarrow f_2^{-1} & & \\
1124 & \xrightarrow{f_{12}} & 111122 & \\
& \swarrow f_2^{-1} & &
\end{array}
\qquad
\begin{array}{lcll}
11114 & \xrightarrow{f_\emptyset} & 11114 & \multirow{3}{*}{$\} \; h^{-1}_{1,0}$} \\
& \swarrow f_1^{-1} & & \\
1124 & \xrightarrow{f_{12}} & 111122 & \\
& \swarrow f_1^{-1} & & \\
11222 & \xrightarrow{f_\emptyset} & 11222 & \} \; h^{-1}_{12,0} \\
& \swarrow f_{12}^{-1} & & \\
224 & \xrightarrow{f_\emptyset} & 224 & \} \; h^{-1}_{2,0} \\
& \swarrow f_2^{-1} & & \\
44 & \xrightarrow{f_\emptyset} & 44 & \} \; h^{-1}_{4,0} \\
& \swarrow f_4^{-1} & & \\
8 & \xrightarrow{f_\emptyset} & 8 &
\end{array}
$$

for a total of 28 steps.

Case 3. Algorithm B.

Consider the Glaisher bijection presented above. Suppose one wanted to write an algorithm (for implemention on a computer) which would produce the same bijection. Suppose $\pi_0 = 11111111$. One could count $l_1 = 8$, find the binary representation of 8 (its a simple "Euclidean-type" algorithm), and then print the result appropriately multiplied by an odd integer. Or one could achieve the same effect by simply identifying repeated parts and rewriting as one part. In the given example, we would find

Step 1 Input: 11111111

Repeated parts? Yes 11 Rewrite as 2

Step 2 New partition 1111112

Repeated parts? Yes 11 Rewrite as 2

Step 3	New partition	111122	
	Repeated parts?	Yes 11	Rewrite as 2
Step 4	New partition	11222	
	Repeated parts?	Yes 22	Rewrite as 4
Step 5	New partition	1124	
	Repeated parts?	Yes 11	Rewrite as 2
Step 6	New partition	224	
	Repeated parts?	Yes 22	Rewrite as 4
Step 7	New partition	44	
	Repeated parts?	Yes 44	Rewrite as 8
Step 8	New partition	8	
	Repeated parts?	No.	Print 8

This is exactly the workings of Algorithm B.

1

In this section, we start with the properties of f maps operating on a partition π; obtain a unique representation of π in terms of a product of f^{-1} maps, and then show that Algorithm B generates a bijection between elements of A_0 with those in B_0.

PROPOSITION 1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be sieve-equivalent families whose defining multisets are pairwise disjoint. If $f_j^{-1} f_i(\pi)$ is consistent then $f_i f_j^{-1}(\pi)$ is consistent and*

$$f_j^{-1} f_i(\pi) = f_i f_j^{-1}(\pi).$$

Proof. We assume

$$f_j^{-1} f_i(\pi) = \pi - a_i + b_i - b_j + a_j$$

is consistent. Since $b_i \cap b_j = \varnothing$ $(i \neq j)$

$$= \pi - a_i - b_j + b_i + a_j$$
$$= \pi - b_j - a_i + b_i + a_j$$
$$= \pi - b_j - a_i + a_j + b_j,$$

and since $a_i \cap a_j = \varnothing$ $(i \neq j)$

$$= \pi - b_j + a_j - a_i + b_i$$
$$= f_i f_j^{-1}(\pi),$$

which is also consistent.

LEMMA 1. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be sieve-equivalent families whose defining multisets are pairwise disjoint. Suppose the map*

$$C(\pi_0) = f_{q(r)}^{\varepsilon(r)} f_{q(r-1)}^{\varepsilon(r-1)} \cdots f_{q(1)}^{\varepsilon(1)}(\pi_0)$$

is consistent for some $\pi_0 \in A_0$. *Then* C *can be reduced to a product of* f_i^{-1} *maps, i.e.,*

$$C(\pi_0) = f_{w(s)}^{-1} f_{w(s-1)}^{-1} \cdots f_{w(1)}^{-1}(\pi_0).$$

Proof. Consider the subsequence of maps $\{f_{q(i_k)}^{\varepsilon(i_k)}\}$ in C with the property that $\varepsilon(i_k) = +1$. We prove by induction that for each $k = 1, 2, ..., p$, the product

$$C^{i_k} = f_{q(i_k)}^{\varepsilon(i_k)} f_{q(i_k-1)}^{\varepsilon(i_k-1)} \cdots f_{q(1)}^{\varepsilon(1)}$$

can be written in the form

$$f_{w(v)}^{-1} f_{w(v-1)}^{-1} \cdots f_{w(1)}^{-1}.$$

Let $k = 1$. Consider

$$C^{i_1}(\pi_0) = f_{q(i_1)}^{+1} f_{q(i_1-1)}^{-1} \cdots f_{q(1)}^{-1}(\pi_0).$$

Since the a_i and b_i multisets are pairwise disjoint, $f_{q(i_1)}$ commutes with every $f_{q(i)}^{-1}$ $(i = i_1 - 1,\ i_1 - 2, ..., 2, 1)$ unless there is an $i^* < i_1$ such that $q(i^*) = q(i_1)$. If there is such an i^*, then we are done: the two maps $f_{q(i_1)}$ and $f_{q(i^*)}^{-1}$ match, their composition is the identity map, and only maps with $\varepsilon(i) = -1$ would be left. So suppose there does not exist an $i^* < i_1$ such that $q(i^*) = q(i_1)$. This implies

$$C^{i_1}(\pi_0) = f_{q(i_1)} f_{q(i_1-1)}^{-1} \cdots f_{q(1)}^{-1}(\pi_0)$$
$$= f_{q(i_1-1)}^{-1} \cdots f_{q(1)}^{-1} f_{q(i_1)}(\pi_0), \qquad (*)$$

i.e., $f_{q(i_1)}$ can be exchanged, in succession, with every $f_{q(i)}^{-1}$ $(i < i_1)$ until it reaches π_0. $C^{i_1}(\pi_0)$ is consistent, hence the product in $(*)$ is consistent also. But $a_{q(i_1)} \not\subseteq \pi_0$ since π_0 contains no $a_{q(i)}$-multisets, hence $f_{q(i_1)}(\pi_0)$ is not consistent. This contradiction establishes the case $k = 1$.

Assume the induction hypothesis is true for the $k-1$st case. We show that it is true for the kth case. Consider

$$C^{i_k}(\pi_0)=f^{+1}_{q(i_k)}f^{\varepsilon(i_k-1)}_{q(i_k-1)}\cdots f^{\varepsilon(1)}_{q(1)}(\pi_0)$$
$$=f_{q(i_k)}f^{-1}_{q(i_k-1)}\cdots f^{-1}_{q(i_{k-1}+1)}[RC^{i_{k-1}}(\pi_0)],$$

where $RC^{i_{k-1}}$ is the reduced product of f_i^{-1} maps from $C^{i_{k-1}}$ guaranteed by the induction hypothesis. The $f^{\varepsilon(i)}_{q(i)}$ maps to the right of $f_{q(i_k)}$ all have $\varepsilon(i)=-1$, and this is the condition under which we proved the case when $k=1$: Hence the same argument establishes the kth case.

The conclusion follows by induction.

LEMMA 2. *Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are sieve-equivalent families whose defining multisets are pairwise disjoint. Fix a partition π.*

(a) *Suppose there exist a $\pi_0 \in A_0$ and a consistent map*

$$C(\pi_0)=f^{-1}_{q(m)}f^{-1}_{q(m-1)}\cdots f^{-1}_{q(1)}(\pi_0)$$

with $\pi=C(\pi_0)$. Then π_0 is unique and the representation of

$$\pi=f^{-1}_{q(m)}\cdots f^{-1}_{q(1)}(\pi_0)=C(\pi_0)$$

is unique up to the order of the $q(i)$'s.

(b) *Suppose there exist $\pi' \in B_0$ and a consistent map*

$$C(\pi)=f^{-1}_{u(v)}f^{-1}_{u(v-1)}\cdots f^{-1}_{u(1)}(\pi)$$

with $\pi'=C(\pi)$. Then π' is unique and $u(1)$, $u(2)$, ..., $u(v)$ are uniquely determined up to order.

Proof. (a) Suppose we have two maps $\tilde{C}$ and C, and two elements $\tilde{\pi}$ and π_0 in A_0 (not necessarily different) with the properties: (i) $C(\pi_0)$ and $\tilde{C}(\tilde{\pi})$ are consistent and (ii) $\pi=\tilde{C}(\tilde{\pi})=C(\pi_0)$, i.e.,

$$\pi=f^{-1}_{s(r)}\cdots f^{-1}_{s(1)}(\tilde{\pi})=f^{-1}_{q(m)}\cdots f^{-1}_{q(1)}(\pi_0). \tag{$*$}$$

Without loss of generality, assume $r \leqslant m$. Since the product on the left-hand side of ($*$) is consistent, we know that the partition, π, contains the $a_{s(r)}$-multiset. Hence the right-hand side also contains an $a_{s(r)}$-multiset, and we can apply an $f_{s(r)}$ map to both sides

$$f^{-1}_{s(r-1)}\cdots f^{-1}_{s(1)}(\tilde{\pi})=f_{s(r)}f^{-1}_{q(m)}\cdots f^{-1}_{q(1)}(\pi_0). \tag{$**$}$$

We perform the same operation r times to get

$$\tilde{\pi}=f_{s(1)}f_{s(2)}\cdots f_{s(r)}f^{-1}_{q(m)}\cdots f^{-1}_{q(1)}(\pi_0).$$

We reduce this expression to a product of f_i^{-1} maps by Lemma 1. If $r<m$, we end with

$$\tilde{\pi}=f_{w(k)}^{-1}\cdots f_{w(1)}^{-1}(\pi_0).$$

But this can never happen, since $\tilde{\pi}$ contains no a_i-multisets and the partition on the right-hand side does ($a_{w(k)}$, for example). Therefore $r=m$, and $\tilde{\pi}=\pi_0$ and the two representations of π are in fact the same up to the order of the $q(i)$'s.

(b) Exchange A_0 with B_0, f_i^{-1} with f_i, and π_0 with π' in the statement of part (a). We get

$$\pi=f_{u(1)}f_{u(2)}\cdots f_{u(v)}(\pi').$$

Apply $f_{u(i)}^{-1}$ maps ($i=1, 2, ..., v$) to both sides.

Algorithm B

[The input is a partition $\pi_0\in A_0$ and the output is a partition $\pi'\in B_0$.]

Input $\pi_0\in A_0$, $\pi'\leftarrow\pi_0$.

Repeat until π' contains no b_i-multiset

$$\pi'\leftarrow\pi'-b_i+a_i$$

End {Repeat}.

PRINT π'.

Exit.

EXAMPLE. Suppose the two lists of multisets are

$$\mathfrak{L}_1=\{\{2\}, \{4\}, \{6\}, \{8\}, ...\}$$
$$\mathfrak{L}_2=\{\{11\}, \{22\}, \{33\}, \{44\}, ...\}.$$

Then A_0 equals the set of partitions containing no even part and B_0 equals the set of partitions containing no repeated part. A bijection between A_0 and B_0 yields Euler's identity: the number of partitions of n whose parts are distinct equals the number of partitions whose parts are odd.

Let $n=10$. We input $\pi_0=111133\in A_0$ into Algorithm B.

π'

111133 But π' contains the b_3-multiset, $\{33\}$,
hence $\pi'\leftarrow 111133-33+6$.

11116 But π' contains the b_1-multiset, $\{11\}$,
hence $\pi'\leftarrow 11116-11+2$.

1126	Exchange $\{11\}$ for $\{2\}$ to get
	$\pi' \leftarrow 1126 - 11 + 2$
226	Exchange $\{22\}$ for $\{4\}$ to get
	$\pi' \leftarrow 226 - 22 + 4$
46	π' contains no b_i-multiset. STOP

THEOREM 1. *Suppose* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are sieve-equivalent families whose defining multisets are pairwise disjoint. Then Algorithm* B *produces a bijection between the elements of* A_0 *with those in* B_0.

Proof. We first show that Algorithm B must stop. Each step in the repeat cycle produces the product

$$C_j(\pi_0) = f_{q(j)}^{-1} f_{q(j-1)}^{-1} \cdots f_{q(1)}^{-1}(\pi_0).$$

Since, by Algorithm B, $C_j(\pi_0)$ is consistent, Lemma 2(a) implies $C_i(\pi_0) = \pi_i \neq \pi_j = C_j(\pi_0)$ for $i \neq j$. Since there are only a finite number of partitions of n, Algorithm B can produce only a finite number of different partitions for each input partition π_0, and therefore must halt after a finite number of steps.

Note that the output partition of Algorithm B is an element of B_0 since it contains no b_i-multisets.

The uniqueness statement Lemma 2(b) shows that the algorithm defines a map from A_0 to B_0 despite the choices which arise in the Repeat Cycle.

The uniqueness statement Lemma 2(a) shows that the map from A_0 to B_0 defined by Algorithm B is one-to-one. We show the map is onto by running the algorithm "backwards," i.e. starting with an input partition $\pi' \in B_0$ and exchanging b_i-multisets for a_i-multisets. By the dual statement of Lemma 2(a), this map is also one-to-one and the proof is complete.

2

How fast is Algorithm B?

Fix n. In the odd-distinct case it's easy to see that if the input partition has a total of k odd parts, then at the end of each cycle of the loop, the new partition has one less part than its predecessor (a repeated part is rewritten as one part). Hence the bijective mate is found in less than k steps. The largest k can be is n (n copies of 1), so the algorithm stops in less than n steps.

3

In this section we show that the output from Algorithm B is the same as the output from both the Garsia–Milne–Remmel algorithm (GMRA) and the Gordon algorithm.

Recall the definition of the f map. In the Gordon algorithm,

$$f_S(\pi) = \pi - \bigcup_{i \in S} a_i + \bigcup_{i \in S} b_i \qquad \text{and} \qquad f_S^{-1}(\pi) = \pi - \bigcup_{i \in S} b_i + \bigcup_{i \in S} a_i.$$

In the GMRA,

$$f(\pi, S) = \left(\pi - \bigcup_{i \in S} a_i + \bigcup_{i \in S} b_i, S\right) \quad \text{and} \quad f_S^{-1}(\pi, A) = \left(\pi - \bigcup_{i \in S} b_i + \bigcup_{i \in S} a_i, S\right).$$

The f maps in both Gordon and GMRA are essentially the same. Both exchange a_i-multisets for b_i-multisets for all $i \in S$. The domain of the f map in GMRA is partition, subset ordered pairs, but since it affects only the first coordinate, we "condense" notation and write $f_S(\pi)$. Let $\{\pi_i\}_i$ represent the sequence of partitions generated by either algorithm in its "travels" from $\pi_0 \in A_0$ to its bijective mate $\pi' \in B_0$. Then π_i is a product of $f_{S(i)}^{\varepsilon(i)}$ maps operating on π_0,

$$\pi_i = f_{S(i)}^{\varepsilon(i)} f_{S(i-1)}^{\varepsilon(i-1)} \cdots f_{S(1)}^{\varepsilon(1)}(\pi_0).$$

Proposition 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be sieve-equivalent families whose defining multisets are pairwise disjoint. Fix $S \subseteq M$, then*

$$f_S(\pi) = f_{q(k)} f_{q(k-1)} \cdots f_{q(1)}(\pi)$$

and

$$f_S^{-1}(\pi) = f_{q(k)}^{-1} f_{q(k-1)}^{-1} \cdots f_{q(1)}^{-1}(\pi),$$

where $f_{q(i)}(\pi) = \pi - a_{q(i)} + b_{q(i)}$ and the indices occurring as subscripts on the f maps can be any rearrangement of the elements of S.

Proof. Expand both $\bigcup_{i \in S} a_i$ and $\bigcup_{i \in S} b_i$ into a sum of pairwise disjoint multisets, then rearrange them appropriately.

Theorem 2. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be sieve-equivalent families whose defining multisets are pairwise disjoint. Then Algorithm* B, *the GMRA, and the Gordon algorithm produce the same bijection between the sets A_0 and B_0.*

Proof. Fix $\pi_0 \in A_0$. Let $\hat{\pi}$ be the bijective mate of π_0 generated by the GMRA

$$\hat{\pi} = C(\pi_0) = f_{S(m)}^{\varepsilon(m)} f_{S(m-1)}^{\varepsilon(m-1)} \cdots f_{S(1)}^{\varepsilon(1)}(\pi_0),$$

where the $f_{S(i)}^{\varepsilon(i)}$ are the maps and sets generated by the GMRA. Note that $C(\pi_0)$ by definition of the GMRA is consistent. We decompose these maps as in Proposition 2 to get

$$\hat{\pi} = f_{q(r)}^{\varepsilon(r)} f_{q(r-1)}^{\varepsilon(r-1)} \cdots f_{q(1)}^{\varepsilon(1)}(\pi_0).$$

By Lemma 1,

$$\hat{\pi} = f_{w(k)}^{-1} f_{w(k-1)}^{-1} \cdots f_{w(1)}^{-1}(\pi_0).$$

Let π' be the bijective mate of π_0 generated by Algorithm B.

$$\pi' = f_{v(s)}^{-1} f_{v(s-1)}^{-1} \cdots f_{v(1)}^{-1}(\pi_0).$$

Since π' and $\hat{\pi}$ are both elements of B_0, the dual statement of Lemma 2(a) implies $\pi' = \hat{\pi}$. The same argument establishes that the output from the Gordon algorithm also agrees with the output from Algorithm B, and the proof is complete.

Remark. Thus we see that the GMRA will produce the same bijection no matter the ordering of the A_i and B_i sets if the defining multisets are pairwise disjoint. This is not the case otherwise.

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